*SIMPLICIAL APPROXIMATION OF FIXED POINTS**

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1. *Introduction.*—In a recent pathbreaking article,[1] Scarf has given a combinatorial theorem and an algorithm related to it for computing approximations of fixed points of continuous mappings of a simplex into itself. Although it is similar in statement to Sperner's lemma, Scarf proved his theorem by a constructive method that is different from the customary proofs of such combinatorial results. Thus, he has opened the possibility of efficient computation for a large class of nonlinear optimization problems that can be cast as fixed-point problems.

It is the purpose of this note to state the precise connection between Scarf's theorem and Sperner's lemma.[2, 4] This connection is contained in a theorem which asserts that the combinatorial structure provided by primitive sets is a special case of that afforded by topological subdivisions. Viewed in the light of this result, Scarf's theorem is a corollary of Sperner's lemma. More important, the theorem has suggested the construction of a number of variations on the basic algorithm, all using Scarf's idea of a systematic search but within the context of simplicial subdivisions. In this note, proofs will only be sketched and but one new algorithm proposed; details of the proofs, the description of other algorithms, and computational experience will be published elsewhere.

2. *Definitions and Notation.*—Let $X = (x_0, x_1, \ldots, x_n)$ denote a point in an $(n + 1)$-dimensional real vector space. Two n-simplices contained in the n-dimensional hyperplane $H = \{X | \sum_k x_k = 1\}$ will appear in the discussion; these are $S = \{X|$ all $x_k \leqq 1\} \cap H$, and $T = \{X|$ all $x_k \geqq 0\} \cap H$. The extreme points of S will be denoted by $X^j = (x_k{}^j)$, where $x_k{}^j = 1 - n$ if k and $x_k{}^j = 1$ otherwise, for $j = 0, 1, \ldots, n$. Following Scarf, we introduce *subdivision points* X^{n+1}, $\ldots, X^N$ chosen arbitrarily within T. As set of $n + 1$ points $(X^{j_0}, X^{j_1}, \ldots, X^{j_n})$ from $P = \{X^0, X^1, \ldots, X^n, X^{n+1}, \ldots, X^N\}$ is called *primitive* if there is no vector $X^j \in P$ such that $x_k{}^j > \min\{x_k{}^{j_0}, x_k{}^{j_1}, \ldots, x_k{}^{j_n}\}$ for all k. In framing these definitions, we have used a different set of points for $X^0, X^1, \ldots, X^n$ than that employed by Scarf to simplify the statement of the theorem of the next section.

To ensure the validity of Scarf's theorem and its connection with the Sperner lemma, it is necessary to make the following assumptions which can be ensured by a perturbation of the subdivision points.

Nondegeneracy Assumption. *For each* $k = 0, 1, \ldots, n$, *the value of* min $\{x_k{}^{j_0}, x_k{}^{j_1}, \ldots, x_k{}^{j_n}\}$ *is achieved by exactly one* $x_k{}^{j_l}$ *for every set of* $n + 1$ *points* $\{X^{j_0}, X^{j_1}, \ldots, X^{j_n}\}$ *from* P.

The *covering simplex* of a set of $n + 1$ points $\{X^{j_0}, X^{j_1}, \ldots, X^{j_n}\}$ from P is defined as

$$\{X|x_k \geqq \min\{x_k^{j_0}, x_k^{j_1}, \ldots, x_k^{j_n}\} \text{ for all } k\}.$$

With the Nondegeneracy Assumption, this set is a closed n-dimensional simplex contained in R. Each of its faces contains exactly one point from the given set, namely that point for which the coordinate corresponding to that face is a minimum. A set of $n + 1$ points is primitive if and only if no $X^j \epsilon P$ lies in the interior of its covering simplex.

3. *Scarf's Theorem and Sperner's Lemma.*—The connection between the concept of primitive sets and the classical topological subdivisions is provided by the following result.

THEOREM. *Let subdivision points $X^{n+1}, \ldots, X^N$ be chosen from T so as to satisfy the Nondegeneracy Assumption. The the family of primitive sets from P defines a pseudomanifold with the simplex $\{X^0, X^1, \ldots, X^n\}$ deleted. If $n = 2$, the family of primitive sets determines a simplicial subdivision of S, in which the simplices are the convex hulls of the primitive sets.*

Recall that a *pseudomanifold* on the points P is a family D of sets of $n + 1$ points from P (called n-simplices) such that if a set of n points is a subset of a set of D, then it is a subset of exactly two sets of D. (Normally, connectedness is also required; this is unimportant in this context.) To state the version of the Sperner lemma that applies here, define a *proper labeling* of a pseudomanifold with the simplex $\{X^0, X^1, \ldots, X^n\}$ deleted as an assignment of a label $l(X) \epsilon \{0, 1, \ldots, n\}$ to each point in P such that $l(X^j) = j$ for $j = 0, 1, \ldots, n$.

SPERNER'S LEMMA. *For every proper labeling of a pseudomanifold with the simplex $\{X^0, X^1, \ldots, X^n\}$ deleted, there exist an odd number of sets in D, each with a complete set of labels $\{0, 1, \ldots, n\}$.*

COROLLARY (SCARF'S THEOREM). *Let subdivision points $X^{n+1}, \ldots, X^N$ be chosen from T so as to satisfy the Nondegeneracy Assumption. Let a label $l(X^j) \epsilon\{0, 1, \ldots, n\}$ be assigned to each $X^j \epsilon P$ with $l(X^j) = j$ for $j = 0, 1, \ldots, n$. Then there exist an odd number of primitive sets, each with a complete set of labels $\{0, 1, \ldots, n\}$.*

Proof: The corollary follows immediately from the theorem above and the version of Sperner's lemma stated here.

4. *Structure of the Algorithms.*—The basic idea behind the algorithms of systematic search for a simplex of the subdivision with a complete set of labels can be explained in graph-theoretical terms. Consider a graph in which each node corresponds to a simplex of the subdivision with $\{0, 1, \ldots, n-1\}$ among its labels. A branch connects two nodes if and only if the intersection of the two associated simplices has exactly the labels $\{0, 1, \ldots, n - 1\}$. We shall assign the nodes of this graph to three classes as follows:

A: the simplex has labels $\{0, 1, \ldots, n - 1\}$ in the boundary of the original simplex.

B: the simplex is not in A *and* has some label repeated.

C: the simplex has labels $\{0, 1, \ldots. n\}$.

We shall denote generic members of A, B, C by α, β, and γ; note that A and C need not be disjoint but that B is disjoint from A and C. Each node of the graph is incident to at most one branch (classes A and C) or exactly two branches (class B). Therefore, the components of the graph fall into four types (where all intermediate nodes, if any, are β's): (*i*) $\alpha \ldots \alpha$; (*ii*) $\alpha \ldots \gamma$, which may degenerate to $\alpha = \gamma$; (*iii*) $\gamma \ldots \gamma$; (*iv*) cycles composed of β's.

Let there be n_1, n_2, and n_3 components of the first three types, respectively. If we let the number of nodes of types A and C be a and c, respectively, we have $2n_1 + n_2 = a$ and $n_2 + 2n_3 = c$, and hence $2(n_1 + n_2 + n_3) = a + c$. *Therefore, a and c have the same parity.*

Within this framework, the algorithm of Daniel I. A. Cohen[3] may be considered as examining the components beginning with a node α until a component terminating with a node γ is found. The initiation of his algorithm requires the availability of all nodes of class A, which is assured by a nonconstructive assumption.

For the pseudomanifolds corresponding to Scarf's primitive sets as given in the preceding section, the set A consists of the single simplex containing $\{X^0, X^1, \ldots, X^{n-1}\}$, and hence if we examine the successive nodes in the component that it initiates, we must terminate at a node γ.

In the algorithm to be proposed in the next section, another method is used to ensure that we have a subdivision with but one node α.

5. *An Algorithm.*—The algorithm to be proposed utilizes a standard subdivision used previously for a proof of a cubical Sperner lemma.[4] If we let $T = \{X | X = (x_0, x_1, \ldots, x_n) \geqq 0, \sum_k x_k = 1\}$ as before, then a subdivision of T is defined by a positive integer D. The vertices of the subdivision are $(k_0/D, \ldots, k_n/D)$, where the k_j are nonnegative integers and $\sum_j k_j = D$. A simplex of the subdivision can always be specified in a very compact form by giving a vertex $X^0 = (k_0^0/D, \ldots, k_n^0/D)$ and a permutation $\pi = (j_1, \ldots, j_n)$ of $(1, \ldots, n)$. The vertices of the simplex are then defined recursively by

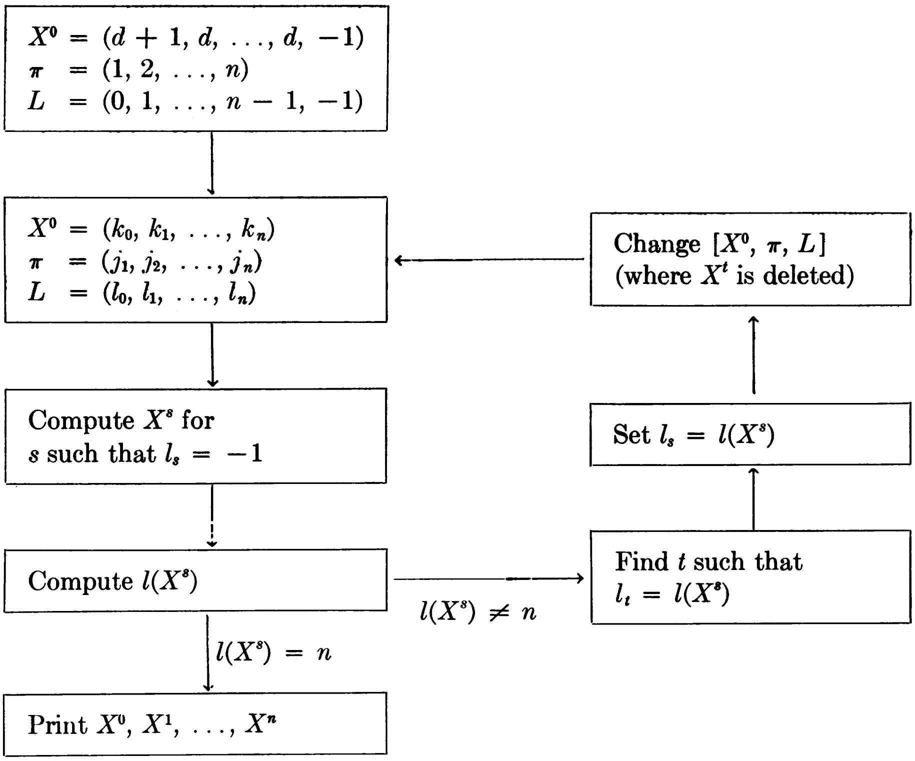
$$DX^t = (k^{t-1}, \ldots, k^{t-1}_{j_t-1} - 1, k^{t-1}_{j_t} + 1, \ldots, k^{t-1}_n)$$

for $t = 1, \ldots, n$.

If we drop one vertex of a simplex of this subdivision, then either there is a unique new vertex that can be introduced to form a new simplex, or the remaining vertices lie in the boundary. The new vertex to be introduced may be calculated by a simple formula.

We assume that the vertices of T have been provided with a *proper labeling* (for T, this means that $l(X) = k$ implies $x_k > 0$). To initiate a systematic search for a simplex with a complete set of labels, we need a unique starting simplex of class A. Since there is no guarantee that such will exist, we provide one by enlarging our original simplex by one layer of the subdivision on all sides and by providing a special labeling on this extra set of vertices. Precisely, we enlarge T to $\bar{T} = \{X | \text{all } x_k \geqq -1 \text{ and } \sum_k x_k = 1\}$ and define $l(X)$ = smallest index l for which $x_l = \max_k\{x_k\}$ if some $x_k < 0$. This device has an obvious geometric motivation that will be provided in the detailed version of this note.

The algorithm is defined by the flow diagram that follows; in the algorithm, d is a positive integer and $D = nd$. At each stage, we keep only a record of X^0 (multiplied by D to keep the data in integers), π, and the vector L of labels of the current simplex specified by X^0 and π.



Flow Diagram of Algorithm

The only part of the flow diagram that may need explanation is the changes that must be made in $[X^0, \pi, L]$ when X^t is deleted. These changes are specified in the table below:

	$X^0 = (k_0, k_1, \ldots, k_n)$ becomes	$\pi = (j_1, \ldots, j_n)$ becomes	$L = (l_0, l_1, \ldots, l_n)$ becomes
$t = 0$	$(k_0, \ldots, k_{j_1-1} - 1, k_{j_1} + 1, \ldots, k_n)$	$(j_2, \ldots, j_n, j_1)$	$(l_1, l_2 \ldots, l_n, -1)$
$0 < t < n$	$(k_0, k_1, \ldots, k_n)$	$(j_1, \ldots, j_{t+1}, j_t, \ldots, j_n)$	$(l_0, \ldots, l_{t-1}, -1 \; l_{t+1}, \ldots, l_n)$
$t = n$	$(k_0, \ldots, k_{j_n-1} + 1, k_{j_n} - 1, \ldots, k_n)$	$(j_n, j_1, \ldots, j_{n-1})$	$(-1, l_0, l_1, \ldots, l_{n-1})$

The advantages of the proposed specification of a simplex and its modification over Scarf's original proposal[1] are the following: (*a*) no problem of degeneracy; (*b*) no search procedure; and (*c*) fixed and small memory requirements. More than a year ago, Terje Hansen[5] discovered the same compact description and associated pivot steps; he has used them in an extremely effective manner for a computer program that uses the same initial α as Scarf's algorithm. Lloyd Shapley[5] has also proposed and programmed a variable-dimension algorithm that uses a compact description of a barycentric subdivision of a simplex.

The current limitations on the problems that can be solved by these algorithms are therefore not in storage but in the total number of iterations. Computational experience will be reported in another article.

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1 Scarf, H., "The approximation of fixed points of a continuous mapping," *SIAM J. Appl. Math.*, **15**, 1328–43 (1967).

2 Sperner, E., "Neuer Beweis für die Invarianz der Dimensionzahl und des Gebietes," *Abh. Math. Sem. Univ. Hamburg*, **6**, 265–72 (1928).

3 Cohen, D. I. A., "On the Sperner Lemma," *J. Combinatorial Theory*, **2**, 585–7 (1967).

4 Kuhn, H. W., "Some combinatorial lemmas in topology," *IBM J. Res. Develop.*, **4**, 518–24 (1960).

5 Private communication.